SOME REMARKS ON THE SOLVABILITY OF SOME ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with the solvability of some abstract differential equation of type $\dot{u}(t) + Au(t) + Bu(t) \ni f(t)$, $t \in (0,T]$, $u(0) = 0$, where A is a linear selfadjoint operator and B is a nonlinear (possibly multi-valued) maximal monotone operator in a real Hilbert space $\mathbb{H}$ with the normalization $0 \in B(0)$. We use the concept of variational sum introduced by H. Attouch, J.-B. Baillon, and M. Théra, to investigate solutions to the given differential equation.

1. Introduction

Our aim in this paper is to investigate on the solvability of some abstract differential equation of type

$$(1.1) \quad \begin{cases} \dot{u}(t) + Au(t) + Bu(t) \ni f(t), \quad t \in (0,T] \\ u(0) = 0 \end{cases}$$

where A is a linear selfadjoint monotone operator and B (the operator B some times equals to $\partial\psi$, the subdifferential of a convex lower semicontinous proper function $\psi : \ \mathbb{H} \mapsto \mathbb{R} \cup +\infty$) is a nonlinear maximal monotone operator (possibly multi-valued) in the real Hilbert space $(\mathbb{H}\ ;\ \langle\ ,\ \rangle)$ with the normalization $0 \in B(0)$. The function f belongs to $L^2(0,T;\mathbb{H})$, where $L^2(0,T;\mathbb{H})$ is the Hilbert space endowed with the inner product

$$(1.2) \qquad \langle\langle u\ ,\ v\rangle\rangle = \int_0^T \langle u(t)\ ,\ v(t)\rangle\, dt$$

It is convenient to write (1.1) of the form

$$(1.3) \qquad \mathcal{S}u + \mathcal{A}u + \mathcal{B}u \ni f$$

where $\mathcal{S}$ is defined in $L^2(0,T;\mathbb{H})$ by

$$\begin{cases} D(\mathcal{S}) = \{u \in \mathbb{H}^1(0,T;\mathbb{H}) \mid u(0) = 0\} \\ \mathcal{S}u = \dot{u}, \quad \forall u \in D(\mathcal{S}) \end{cases}$$

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The operator $\mathcal{A}$ is defined in $L^2(0,T;\mathbb{H})$ by: $u \in D(\mathcal{A})$ and $\mathcal{A}u = v$ iff $u, v \in L^2(0,T;\mathbb{H})$ and $Au(.) = v(.)$. In the same way, the operator $\mathcal{B}$ is defined in $L^2(0,T;\mathbb{H})$ by: $u \in D(\mathcal{B})$ and $\mathcal{B}u = v$ iff $u, v \in L^2(0,T;\mathbb{H})$ and $\mathcal{B}u(.) = v(.)$.

It is well-known that the corresponding operators $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{B}$ are maximal monotone on $L^2(0,T;\mathbb{H})$, see, e.g., [24].

Recall that various types of equation (1.1) have been investigated in the past decades by several mathematicians, see, e.g., [2, 6, 8, 13, 14, 22, 23, 24, 25, 26, 27].

This paper is concerned with the solvability to the equation (1.1). Assuming that A is a linear, selfadjoint monotone operator (possibly unbounded) and that the operator B (possibly multi-valued) is maximal accretive. Under suitable assumptions, we shall establish the solvability of the equation (1.1). Recall that the innovating idea in this paper is the use of the powerful concept of the variational sum, introduced by H. Attouch, J B. Baillon, and M. Théra in [4].

In what follows, we assume that A is a linear, self-adjoint monotone operator (possibly unbounded) and that B is maximal monotone(possibly multi-valued), with the normalization $0 \in B(0)$. Instead of considering (1.1), we consider the solvability of (1.3).

Recall that the Moreau-Yosida approximation of $\mathcal{B}$ is defined as follows

$$\mathcal{B}_\lambda = \frac{1}{\lambda}\left[I - (I + \lambda\mathcal{B})^{-1}\right] \quad \lambda > 0$$

The Moreau-Yosida approximation $\mathcal{B}_\lambda, \quad \lambda > 0$ is an everywhere defined, Lipschitz continuous, and maximal monotone operator, see, e.g., [7]. Also, note that $\mathcal{B}_\lambda$ is single-valued and that $\mathcal{B}_\lambda \subset \mathcal{B}\ (I + \lambda\mathcal{B})^{-1}$ in the sense of the corresponding graphs.

Now recall the definition and some details about the concept of the variational sum $(\mathcal{A} + \mathcal{B})_v$ of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{F}$ be the *filter* of all the pointed neighborhoods of the origin $(0,0)$ in the set $I = \{(\lambda, \mu) \in \mathbb{R}^2 : \quad \lambda,\ \mu \geq 0, \quad \lambda + \mu \neq 0\ \}$ and $\liminf_{\mathcal{F}}$ for $\lim_{\lambda \to 0,\ \mu \to 0,\ (\lambda,\mu) \in I}$. The variational sum of the maximal monotone operators $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$(\mathcal{A}\ +\ \mathcal{B})_v = \liminf_{\mathcal{F}}\ (\mathcal{A}_\lambda + \mathcal{B}_\lambda) \tag{1.4}$$

where the limit Inferior is understood in the sense of Kuratowski-Painlevé, when $\mathcal{A}$ and $\mathcal{B}$ are identified with their graphs. The equation (1.4) can be equivalently expressed in terms of resolvents, that is, for any $w \in L^2(0,T;\mathbb{H})$, the family $\{u_{\lambda,\mu} : \quad (\lambda,\mu) \in I\}$ of solutions of

$$u_{\lambda,\mu} + \mathcal{A}_\lambda u_{\lambda,\mu} + \mathcal{B}_\mu u_{\lambda,\mu} u_{\lambda,\mu}\ \ni w \tag{1.5}$$

converges (with respect to the *filter* $\mathcal{F}$) to the solution u of

$$u + (\mathcal{A} + \mathcal{B})_v u \ni w \tag{1.6}$$

More details about the concepts of the algebraic sum, variational sum, generalized sum or the extended sum can be found in [3, 4, 7, 8, 14, 16, 17, 18, 28, 29, 30]. Nevertheless, recall that among the main motivations of the concept of variational sum, we have the fact that the algebraic sum of two operators in not always well-adapted to problems arising in mathematics, see [18, 19, 20], for examples.

2. Existence of solutions

Consider the equation (1.3) in the Hilbert space $L^2(0,T;\mathbb{H})$. Thus, the existence problem of solutions to (1.3) is equivalent to finding conditions for which the algebraic sum $\mathcal{A} + B$ is a maximal accretive operator in $L^2(0,T;\mathbb{H})$. As stated in the introduction, the algebraic sum is not well-adapted to the present situation. To overcome such a difficulty, we shall deal this the variational sum $(\mathcal{A}+\mathcal{B})_v$ of $\mathcal{A}$ and $\mathcal{B}$ and compute it.

2.1. **Revolvent Commuting case.** Let $\mathcal{A}$ and $\mathcal{B}$ be the operators described above, where $\mathcal{B}$ is supposed to be a linear operator. Assume that they commute in the sense of resolvent, that is,

$$(I+\lambda\mathcal{A})^{-1}(I+\mu\mathcal{B})^{-1} = (I+\mu\mathcal{B})^{-1}(I+\lambda\mathcal{A})^{-1}, \ \ \forall\lambda,\mu>0 \tag{2.1}$$

In this case, it is well-known that the algebraic sum $\mathcal{A}+\mathcal{B}$ is closable, and by a result of Da Prato and P. Grisvard (see [16]), we also know $\overline{\mathcal{A}+\mathcal{B}}$ is m-accretive.

We have

Proposition 2.1. Let $\mathcal{A}$, $\mathcal{B}$ be the corresponding operators to A and B described above. Assume that $\mathcal{B}$ is linear and that (2.1) holds; then the problem (1.3) has a unique solution.

Proof. Since $\mathcal{A}$ (densely defined) and $\mathcal{B}$ are m-accretive satisfying (2.1). Then according to [16], $-\overline{\mathcal{A}+\mathcal{B}}$ is m-dissipative. Therefore the Hille-Yosida theorem guarantees the existence of a unique solution to (1.3). Since (1.1) and (1.3) are equivalent, then so does (1.1). ☐

Remark 2.2. Considering the given linear m-accretive operators $\mathcal{A}$ and $\mathcal{B}$ described above. If one of them generates an analytic semigroup and that (2.1) holds, it is also known that $-\overline{\mathcal{A}+\mathcal{B}}$ is m-dissipative. This is in fact a consequence of a result due to Dore and Venni (see [21]).

2.2. **Case where $\mathcal{B} = \partial\psi$.** We assume that the operator $\mathcal{B} = \partial\phi$ is the subdifferential of a convex semicontinuous proper function $\psi : \ L^2(0,T;\mathbb{H}) \mapsto \mathbb{R} \cup +\infty$. Under previous assumptions, it is well-known that the Moreau-Yosida approximation $\mathcal{B}_\lambda = (\partial\psi)_\lambda = \partial\psi_\lambda$ where

$$\psi_\lambda(x) = \inf_{v\in\mathbb{H}}\{\psi(v) + \frac{1}{2\lambda}\|x-v\|^2\}$$

Recall that since $\mathcal{A}$ is a self-adjoint monotone operator, then it can be expressed as $\mathcal{A} = \partial\phi$, the subdifferential of the convex semicontinuous proper functional $\phi : \ L^2(0,T;\mathbb{H}) \mapsto \mathbb{R} \cup +\infty$ defined by

$$(2.2) \quad \phi(u) = \begin{cases} \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|^2 & \text{if } u \in D(\mathcal{A}^{\frac{1}{2}}) \\ +\infty & \text{elsewhere} \end{cases}$$

Since $\mathcal{B}_\lambda$ is m-accretive and Lipschitz $\forall\lambda > 0$, it is well-known that $\mathcal{A} + \mathcal{B}_\lambda$ is maximal monotone operator, see, e.g.,[7]. Therefore, for all $w \in L^2(0,T;\mathbb{H})$, the equation

$$(2.3) \qquad u_\lambda + \mathcal{A}u_\lambda + \mathcal{B}_\lambda u_\lambda = w, \quad \lambda > 0$$

has a unique solution $u_\lambda \in D(\mathcal{A})$. We also know that there exists a unique $u \in D((\mathcal{A}+\mathcal{B})_v)$ such that u_λ converges to u as λ goes to zero, and

$$(2.4) \qquad w \ \in \ u + (\mathcal{A}+\mathcal{B})_v u$$

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be the operators described above such that $\mathcal{A} = \partial\phi$ and $\mathcal{B} = \partial\psi$ on $L^2(0,T;\mathbb{H})$. Assume that $D(\phi) \cap D(\psi) \neq \varnothing$. Then the variational sum of $\mathcal{A}$ and $\mathcal{B}$ is given by*

$$(\mathcal{A}+\mathcal{B})_v = \partial(\phi+\psi)$$

Proof. This is a consequence of (Theorem 7.2, [4]), it is straightforward. $\square$

Corollary 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be the operators described above such that $\mathcal{A} = \partial\phi$ and $\mathcal{B} = \partial\psi$ on $L^2(0,T;\mathbb{H})$. Assume that $D(\phi) \cap D(\psi) \neq \varnothing$. Then the problem given by*

$$(2.6) \qquad \dot{u} + \partial(\phi+\psi)u \ni f$$

has a unique solution.

Remark 2.5. In the case where $\overline{\mathcal{A}+\mathcal{B}}$ is an m-accretive operator then, $\partial(\phi+\psi) = \overline{\mathcal{A}+\mathcal{B}}$. As a consequence (1.1) has a unique solution.

Let $\mathcal{A}$ be selfadjoint monotone operator and let $\mathcal{B}$ be a nonlinear maximal monotone operator (possibly multi-valued) on $(L^2(0,T;\mathbb{H}) \ ; \ \langle\langle \ , \ \rangle\rangle)$. They said forming an acute angle if the following holds:

$$(2.7) \qquad ((\mathcal{A}_\lambda u, \mathcal{B}_\mu u \ )) \ \geq \ 0, \ \ \forall\lambda,\mu > 0, \ \ u \in L^2(0,T;\mathbb{H})$$

Theorem 2.6. *Let $\mathcal{A}$ be a linear self-adjoint monotone operator and let $\mathcal{B}$ be a nonlinear maximal monotone operator (possibly multi-valued). Assume that (2.7) holds. Then $(\mathcal{A}+\mathcal{B})_v \equiv \mathcal{A}+\mathcal{B}$ is an m-accretive nonlinear operator on $L^2(0,T;\mathbb{H})$.*

Proof. Since $\mathcal{A}+\mathcal{B}_\mu$ is m-accretive, then for all $w \in L^2(0,T;\mathbb{H})$, there exists a unique $(u_\mu)_{\mu>0} \in D(\mathcal{A})$ such that

$$\mu u + \mathcal{A}u_\mu + \mathcal{B}_\mu u_\mu = w, \quad \mu > 0 \tag{2.8}$$

According to Brézis-Grandall-Pazy (see [8]), the problem

$$u + \mathcal{A}u + \mathcal{B}u \ni w \tag{2.9}$$

has a unique solution $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ if and only if the family $(\mathcal{B}_\mu)_{\mu>0}$ is bounded in $L^2(0,T;\mathbb{H})$. Now, a sufficient condition for $(\mathcal{B}_\mu)_{\mu>0}$ to be bounded is guaranteed by (2.7).

Since $(\mathcal{B}_\mu)_{\mu>0}$ is bounded, this implies that u_μ strongly converges to some $u \in L^2(0,T;\mathbb{H})$ and that $\mathcal{B}_\mu u_\mu$ weakly converges to Z as μ goes to 0. Using the fact $\mathcal{B}$ is a maximal monotone operator, it easily follows that $u \in D(\mathcal{B})$ and that $\mathcal{B}u = Z$. Using a similar argument, it turns out that $u \in D(\mathcal{A})$ and that u satisfies (2.9), that is $(\mathcal{A}+\mathcal{B})_v \equiv \mathcal{A}+\mathcal{B}$ is a nonlinear maximal monotone operator on $L^2(0,T;\mathbb{H})$ □

Corollary 2.7. *Let $\mathcal{A}$ be a linear self-adjoint monotone operator and let $\mathcal{B}$ be a nonlinear maximal monotone operator (possibly multi-valued). Assume that (2.7) holds. Then the problem (1.1) has a unique solution.*

2.3. **Examples.**

2.3.1. *Example 1.* Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary $\partial\Omega$. Consider the partial differential equations of type

$$\begin{cases} u_t + Au(t,x) + Bu(t,x) \ni f(t,x), & \forall (t,x) \in (0,T] \times \Omega \\ u(0,x) = 0, & \forall x \in \partial\Omega \end{cases} \tag{2.10}$$

where $A = -\Delta$ and $Bu = F(u)$ with the normalization $F(0) \ni 0$ in $L^2(\Omega)$. The function f belongs to $L^2(0,T;L^2(\Omega))$, where $L^2(0,T;L^2(\Omega))$ is the Hilbert space endowed with the inner product

$$\langle\langle u\ ,\ v\rangle\rangle = \int_0^T \langle u(t,x)\ ,\ v(t,x)\rangle_{L^2(\Omega)} dt$$

It is convenient to write (2.10) of the form

$$\mathcal{S}u + \mathcal{A}u + \mathcal{B}u \ni f \tag{2.11}$$

where

$$\begin{cases} D(\mathcal{S}) = \{u \in \mathbb{H}^1(0,T; L^2(\Omega)) \mid u(0,x) = 0,\ \forall x \in \partial\Omega\} \\ \mathcal{S}u = u_t, \quad \forall u \in D(\mathcal{S}) \end{cases}$$

Recall that A and B are respectively given in the Hilbert space $L^2(\Omega)$ by

$$D(A) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega) \quad \text{and} \quad Au = -\Delta u, \ \ \forall u \in D(A)$$

and

$$\begin{cases} D(B) = \{u \in L^2(\Omega)) \mid F(u) \in L^2(\Omega))\} \\ Bu(x) = F(u)(x) \ \text{ a.e } \ u \in D(B) \end{cases}$$

and that $\mathcal{A}$ and $\mathcal{B}$ are defined in the Hilbert spce $L^2(0,T; L^2(\Omega))$ as described in the introduction.

We assume that $F: \mathbb{R} \mapsto \mathbb{R}$ is everywhere defined non-decreasing function of class C^1 satisfying the assumption $F(0) = 0$. Under such a assumption, then the operator B is m-accretive on $L^2(\Omega)$ (see [Proposition 2.7, [7]]. Thus, the corresponding operators $\mathcal{A}$ and $\mathcal{B}$ are respectively m-accretive operators on $L^2(0,T; L^2(\Omega))$.

We have

Proposition 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be the operators described above. Assume that $F: \mathbb{R} \mapsto \mathbb{R}$ is everywhere defined non-decreasing function of class C^1 satisfying the assumption $F(0) = 0$ and that $D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$. Then the variational sum of $\mathcal{A}$ and $\mathcal{B}$ is a maximal monotone operator and that*

$$(\mathcal{A} + \mathcal{B})_v = \mathcal{A} + \mathcal{B}$$

Proof. It is sufficient to prove that the problem given as

$$\begin{cases} \mu u - \Delta u + F(u) = f \quad \in \Omega, \ \ \mu > 0 \\ u = 0 \quad \text{ on } \ \partial\Omega \end{cases} \tag{2.12}$$

has a unique solution for every $f \in L^2(\Omega)$.

The existence and the uniqueness of a solution to (2.12) is guaranteed by a result of Brézis - Grandall - Pazy (see [Theorem 3.1, [8]]), which says that, under previous assumptions one has

$$\int_\Omega -\Delta u(x) B_\lambda u(x) dx \ \geq \ 0, \ \ \forall u \in \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega), \ \ \lambda > 0$$

That is, $-\Delta + B$ is a maximal monotone operator. Therefore $\mathcal{A} + \mathcal{B}$ is also a maximal monotone operator. According to the definition of the variational sum of $\mathcal{A}$ and $\mathcal{B}$, clearly $(\mathcal{A} + \mathcal{B})_v = \mathcal{A} + \mathcal{B}$ is a maximal monotone operator. □

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be the operators described above. Assume that $F : \mathbb{R} \mapsto \mathbb{R}$ is everywhere defined non-decreasing function of class C^1 satisfying the assumption $F(0) = 0$ and that $D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$. Then the problem (2.10) has a unique solution.*

Proof. The problem (2.10) is equivalent to the problem (2.11). According to Proposition 2.8, the solvability of (2.11) is established. □

2.3.2. *Example 2.* Consider the problem (2.10) where both A and B are linear operators on $\mathbb{H} = L^2(\mathbb{R}^n)$ defined as

$$D(A) = \mathbb{H}^2(\mathbb{R}^n) \quad \text{and} \quad Au = -\Delta u, \ \ \forall u \in D(A)$$

and

$$\begin{cases} D(B) = \{u \in L^2(\mathbb{R}^n) \ | \ Q(x)u \in L^2(\mathbb{R}^n)\} \\ \qquad\qquad Bu = Q(x)u \ \ u \in D(B) \end{cases}$$

where the potential Q satisfies the following assumptions

$$Q(x) > 0, \quad Q \in L^1(\mathbb{R}^n), \ \ \text{and} \ \ Q \notin L^2_{loc}(\mathbb{R}^n) \tag{2.13}$$

Proposition 2.10. Under assumption (2.13), then $D(A) \cap D(B) = \{0\}$.

Proof. The proof of this proposition depends on the dimensional space n (This is explained by the Sobolev embedding) . We will provide the proof in the case where $n \leq 3$. Indeed the proof in the case where $n \geq 4$ can be found in [Proposition 2.1, [20]].

Let $u \in D(-\Delta) \cap D(Q)$ and assume that $u \not\equiv 0$. Since $u \in \mathbb{H}^2(\mathbb{R}^n)$ where $n \leq 3$, then u is a continuous function by Sobolev theorem (see [1]). There exists an opren subset Ω of $\mathbb{R}^n$ and there exists $\delta > 0$ such that $|u(x)| > \delta$ for all $x \in \Omega$. Let Ω' be a compact subset of Ω, equipped with the induced topology by Ω (Ω' is a compact subset of $\mathbb{R}^n$). It easily follows that

$$|Q|_{|\Omega'} = \frac{(|Qu|)_{|\Omega'}}{|u|_{|\Omega'}} \in L^2(\Omega')$$

since $(|Qu|)_{|\Omega'} \in L^2(\Omega')$ and $\frac{1}{|u|_{|\Omega'}} \in L^\infty(\Omega')$. Therefore $Q \in L^2(\Omega')$; this is impossible according to the assumption (2.13), then $u \equiv 0$. □

Example of potential Q satisfying (2.13). Let Ω be a compact subset of $\mathbb{R}^n$ and let G be a complex function satisfying, $\Re e\, G > 0$, $G \in L^1(\Omega)$, $G \notin L^2(\Omega)$ and $G \equiv 0$ on $\mathbb{R}^n - \Omega$. Consider the following rational sequence $\alpha_k = (\alpha_k^1, \alpha_K^2, ..., \alpha_k^n) \in \mathbb{Q}^n$. Then the function Q given by,

$$Q(x) = \sum_{k=1}^{+\infty} \frac{G(x - \alpha_k)}{k^2},$$

satisfies the assumption (2.13).

Note that A and B given above are respectively self-adjoint operators on $L^2(\mathbb{R}^n)$. In what follows, we shall assume that $D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in $L^2(\mathbb{R}^n)$; one can show that the variational sum $(A+B)_v$ and the sum form $A \oplus B$ of A and B coincide, that is,

$$(A+B)_v \equiv A \oplus B$$

Consider the closed sesquilinear forms given by

$$\Phi(u,v) = \int_{R^n} \nabla u \ \bar{\nabla} v dx \ \text{ for all } \ u, v \ \in \mathbb{H}^1(\mathbb{R}^n),$$

$$\Psi(u,v) = \int_{\mathbb{R}^n} Q u \bar{v} dx \ \text{ for all } u, v \ \in D(B^{\frac{1}{2}}),$$

and the sum of the forms Φ and Ψ is given by, $\Upsilon = \Phi + \Psi$, in other words, $\Upsilon(u,v) = \int_{\mathbb{R}^n} (\nabla u \bar{\nabla} v + Q u \bar{v}) dx$ for all $u, v \in D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$. The sesquilinear form Υ is a closed sectorial and densely defined form as sum of closed sectorial and densely defined forms, then there exists a unique m-sectorial operator $A \oplus B$, called sum form of A and B associated to Υ (see [17]) and Υ has the following represention,

$$\Upsilon(u,v) = \langle (A \oplus B) u \ , \ v \rangle \ \text{ for all } u \ \in D(A \oplus B), \ v \ \in D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$$

According to the author ([17]), the operator $A \oplus B$ verifies the well-known condition of Kato, in other words,

$$D((A \oplus B)^{\frac{1}{2}}) = D(\Upsilon) = D(((A \oplus B)^*)^{\frac{1}{2}})$$

The operator $A \oplus B$ has been computed by H. Brézis and T. Kato in [11]. It is given by

$$\begin{cases} D(A \oplus B) = \{ u \in \mathbb{H}^1(\mathbb{R}^n) \ | \ Q|u|^2 \in L^1(\mathbb{R}^n), \quad -\Delta u + Q u \in L^2(\mathbb{R}^n) \} \\ (A \oplus B) u = -\Delta u + Q u \end{cases}$$

Clearly $(A+B)_v \equiv (A \oplus B)$.

Now, since $D(A) \cap D(B) = \{0\}$ under (2.13), the problem (2.10) does not make sense anymore. Another alternative is to consider the following problem

$$\begin{cases} u_t + (A \oplus B) u(t,x) = f(t,x), \ \ \forall (t,x) \in (0,T] \times \Omega \\ u(0,x) = 0, \ \ \forall x \in \partial\Omega \end{cases} \tag{2.14}$$

Clearly, the problem (2.14) has a unique solution.

Let us define $\mathcal{A} \oplus \mathcal{B} = (\mathcal{A} + \mathcal{B})_v$. Let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding operators described in the introduction and defined in the Hilbert space $L^2(0,T; L^2(\mathbb{R}^n))$. The corresponding operator to the sum form $\mathcal{A} \oplus \mathcal{B}$ is defined on $L^2(0,T; L^2(\mathbb{R}^n))$ by: $u \in D(\mathcal{A} \oplus \mathcal{B})$ iff $u \in L^2(0,T; \mathbb{H}^1(\mathbb{R}^n))$, $\ Q|u|^2 \in$

$L^2(0,T;L^1(\mathbb{R}^n))$, and $-\Delta u + Qu \in L^2(0,T;L^2(\mathbb{R}^n))$ with $(\mathcal{A}\oplus\mathcal{B})u = -\Delta u + Qu$ ■

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